8 February 2012, 9:00 – 12:00

Rijksuniversiteit Groningen
Statistiek

Hertentamen

RULES FOR THE EXAM:

- The use of a normal, non-graphical calculator is permitted.
- This is a CLOSED-BOOK exam.
- At the end of the exam you can find a normal table and a chi-squared table.

1. **Point estimation (1).** Let $X_1, \ldots, X_n$ be a sample of independent, identically distributed random variables, with density

$$f_\theta(x) = \begin{cases} \frac{2x}{\theta^2} & 0 < x < \theta \\ 0 & \text{elsewhere} \end{cases}$$

   (a) Determine the maximum likelihood estimator (MLE) of θ. [5 Marks] $\hat{\theta}_n = X_n$

   (b) Let $\hat{\theta}_n$ be the estimator of θ you derived in (a), $E[\hat{\theta}_n]$

      i. Determine whether $\hat{\theta}_n$ is unbiased. [5 Marks]

      ii. Determine whether $\hat{\theta}_n$ is consistent. [5 Marks]

      iii. Determine whether $\hat{\theta}_n$ is sufficient. [5 Marks]

   (c) Why doesn't the Cramer-Rao lower bound apply to unbiased estimates of θ for this distribution? [5 Marks]

2. **Point estimation (2).** Let $X_1, \ldots, X_n$ be a sample of independent, identically distributed $\text{Exp}(\theta)$ random variables, with density

$$f_\theta(x) = \frac{e^{-x/\theta}}{\theta}, \quad x > 0.$$

   (a) Determine the Cramer-Rao lower bound for the variance of an unbiased estimate of θ. [5 Marks]

(b) Determine the MLE $\hat{\theta}$ of θ. [5 Marks]

(c) Check that $\hat{\theta}$ is unbiased an whether it attains the Cramer-Rao lower bound. [5 Marks]

(d) In fact, we are told that $n = 3$ and that the data are given as

$$x_1 = 1, \quad x_2 = 2.5, \quad x_3 = 5.5.$$

i. Determine an approximate 95% confidence interval based on the asymptotic normality of $\hat{\theta}$. [5 Marks]

ii. Determine an approximate 95% confidence interval based on the asymptotic distribution of the deviance $D(\theta)$. (The solution does not have to be exact and you can use Figure 1, but pay attention to the definition of be g!) [5 Marks]

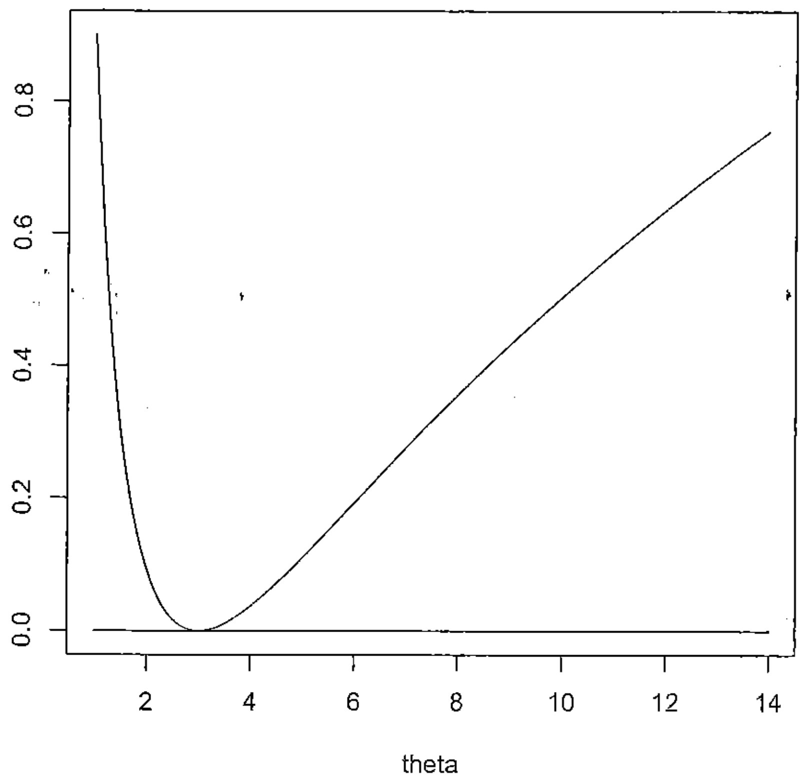

Figure 1: This is the function $g(\theta) = \frac{3}{\theta} + \log(\theta) - \log 3 - 1$.

3. **Optimal testing.** An Atomic Energy Agency is worried that a particular nuclear plant is not managed carefully. It has decided to monitor the plant for the number of days T between successive "minor problems". It performs a single observation T, assuming $T \sim$ Geometric(p), i.e.

$$p_T(k) = (1-p)^{k-1}p, \quad k = 1, 2, \ldots.$$

The management of the plant claims that $p = 0.001$. The law states that the plant should be closed down if $p = 0.003$. The Atomic Energy Agency decides to test the following hypotheses:

$$H_0 : \quad p = 0.001$$
$$H_1 : \quad p = 0.003$$

The optimal critical region is given in the form as $\{t \mid L_0(t)/L_1(t) \leq k\}$.

(a) Determine the cumulative distribution function of T given a general p. [Hint: $\sum_{j=a}^{b} q^j = \frac{q^a - q^{b+1}}{1-q}$ for $|q| < 1$.] [5 Marks]

(b) Determine the critical region and the value k. [15 Marks]

(c) What is the power of this test? [5 Marks]

(d) The next minor incident happens after 36 days. Should the Atomic Energy Agency close down this plant? Yes/No answer is not enough. Give a formally correct argument. [5 Marks]

Statistical tables which may be used in the calculations.

$\nu \setminus \alpha$	0.995	0.99	0.975	0.95	0.05	0.025	0.01	0.005
1	0.000	0.000	0.001	0.004	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	11.070	12.833	15.086	16.750
10	2.156	2.558	3.247	3.940	18.307	20.483	23.209	25.188

Table 1: Values of $\chi^2_{\alpha,\nu}$: the entries in the table correspond to values of x, such that $P(\chi^2_\nu > x) = \alpha$, where χ^2_ν correspond to a chi-squared distributed variable with ν degrees of freedom.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.000	0.004	0.008	0.012	0.016	0.020	0.024	0.028	0.032	0.036
0.1	0.040	0.044	0.048	0.052	0.056	0.060	0.064	0.067	0.071	0.075
0.2	0.079	0.083	0.087	0.091	0.095	0.099	0.103	0.106	0.110	0.114
0.3	0.118	0.122	0.126	0.129	0.133	0.137	0.141	0.144	0.148	0.152
0.4	0.155	0.159	0.163	0.166	0.170	0.174	0.177	0.181	0.184	0.188
0.5	0.191	0.195	0.198	0.202	0.205	0.209	0.212	0.216	0.219	0.222
0.6	0.226	0.229	0.232	0.236	0.239	0.242	0.245	0.249	0.252	0.255
0.7	0.258	0.261	0.264	0.267	0.270	0.273	0.276	0.279	0.282	0.285
0.8	0.288	0.291	0.294	0.297	0.300	0.302	0.305	0.308	0.311	0.313
0.9	0.316	0.319	0.321	0.324	0.326	0.329	0.331	0.334	0.336	0.339
1.0	0.341	0.344	0.346	0.348	0.351	0.353	0.355	0.358	0.360	0.362
1.1	0.364	0.367	0.369	0.371	0.373	0.375	0.377	0.379	0.381	0.383
1.2	0.385	0.387	0.389	0.391	0.393	0.394	0.396	0.398	0.400	0.401
1.3	0.403	0.405	0.407	0.408	0.410	0.411	0.413	0.415	0.416	0.418
1.4	0.419	0.421	0.422	0.424	0.425	0.426	0.428	0.429	0.431	0.432
1.5	0.433	0.434	0.436	0.437	0.438	0.439	0.441	0.442	0.443	0.444
1.6	0.445	0.446	0.447	0.448	0.449	0.451	0.452	0.453	0.454	0.454
1.7	0.455	0.456	0.457	0.458	0.459	0.460	0.461	0.462	0.462	0.463
1.8	0.464	0.465	0.466	0.466	0.467	0.468	0.469	0.469	0.470	0.471
1.9	0.471	0.472	0.473	0.473	0.474	0.474	0.475	0.476	0.476	0.477
2.0	0.477	0.478	0.478	0.479	0.479	0.480	0.480	0.481	0.481	0.482
2.1	0.482	0.483	0.483	0.483	0.484	0.484	0.485	0.485	0.485	0.486
2.2	0.486	0.486	0.487	0.487	0.487	0.488	0.488	0.488	0.489	0.489
2.3	0.489	0.490	0.490	0.490	0.490	0.491	0.491	0.491	0.491	0.492
2.4	0.492	0.492	0.492	0.492	0.493	0.493	0.493	0.493	0.493	0.494
2.5	0.494	0.494	0.494	0.494	0.494	0.495	0.495	0.495	0.495	0.495
2.6	0.495	0.495	0.496	0.496	0.496	0.496	0.496	0.496	0.496	0.496
2.7	0.497	0.497	0.497	0.497	0.497	0.497	0.497	0.497	0.497	0.497
2.8	0.497	0.498	0.498	0.498	0.498	0.498	0.498	0.498	0.498	0.498
2.9	0.498	0.498	0.498	0.498	0.498	0.498	0.498	0.499	0.499	0.499
3.0	0.499	0.499	0.499	0.499	0.499	0.499	0.499	0.499	0.499	0.499

Table 2: Standard Normal Distribution. This means that values in the table correspond to probabilities $P(0 < Z \leq z)$, where Z is a standard normal distributed variable.